PMM U.S.S.R.,Vol.48,No.6,pp. 683-688,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON THE STABILITY OF MIXED CONVECTIVE MOTIONS IN A VERTICAL LAYER WITH WAVE-LIKE BOUNDARIES*

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The stability of the stationary and oscillatory convective motions which develop in a vertical layer with periodically curved boundaries is studied for the case of longitudinal fluid injection. The amplitude of the boundary undulations and the flow of fluid along the layer are both assumed to be small, and methods of perturbation theory are used. The characteristic properties of the incremental spectrum of the spatially periodic motions are studied and the most dangerous types of perturbations as well as the forms of the stability regions are determined.

Theoretical investigations of the effect of spatial inhomogeneity of the boundary conditions on the stability of convection were sparse, and they deal mainly with horizontal layers of fluid /1-3/. Stationary, spatially periodic motions in a vertical layer with curved boundaries were investigated in /4/ for the case of free convection (when the flow was closed), and their stability was investigated in /5/. It was established that the presence of a small but finite flow of fluid along the layer leads to an increase in the number of different modes of flow, and to the appearance of non-stationary convective motions in the region near the threshold.

1. Consider a two-dimensional flow of fluid in an infinite vertical layer at whose solid boundaries

$$x = \pm d\left(1 - \eta\cos\frac{k_0 y}{d}\right)$$

different constant temperatures $T = \pm\Theta$ are maintained. The flow of a fluid across the transverse cross-section is equal to Q. The system of convection equations has the following form in dimensionless variables:

$$\frac{\partial\Delta\psi}{\partial t} = \Delta^2\psi + D(\psi, \Delta\psi) - G\frac{\partial T}{\partial x} \tag{1.1}$$

$$\frac{\partial T}{\partial t} = \frac{1}{P}\Delta T + D(\psi, T)$$

$$D(\psi, F) = \frac{\partial\psi}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial F}{\partial x}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Here ψ is the stream function, T is temperature, G is the Grashof number, and P is the Prandtl number. The boundary conditions are

$$x = -(1 - \eta\cos k_0 y), \quad T = -1, \quad \psi = q, \frac{\partial\psi}{\partial x} = 0 \tag{1.2}$$

$$x = 1 - \eta\cos k_0 y, \quad T = 1, \quad \psi = \frac{\partial\psi}{\partial x} = 0$$

$$y \to \pm\infty, \ |\psi| < \infty, \ |T| < \infty$$

where $q = Q/\nu$ denotes the dimensionless fluid flow.

2. When $\eta = q = 0$, problem (1.1), (1.2) has the solution

$$U_0 = (\psi_0, T_0), \quad \psi_0 = \frac{G}{24}(1 - x^2)^2, \quad T_0 = x \tag{2.1}$$

corresponding to plane parallel flow. When the values of the Prandtl number P are moderate, the solution is monotonically unstable with respect to parturbations with wave number k, when the threshold Grashof number $G_0(k)$ is exceeded. The neutral curve $G_0(k)$ has a minimum $G = G_c$ at some $k = k_c$ /7/. In the case when $q \neq 0$, the instability of the plane parallel flow is oscillatory, and for small q /8/.

Below we study the convection modes for values of G close to G_c, which are realized when q and η are small, but finite. The method of expansion in amplitude of the non-plane parallel

*Prikl.Matem.Mekhan.,48,6,935-941,1984

component ε of the motion used earlier /5/ for the case when $q = 0$, is employed. We introduce the following coordinate transformation which straightens the layer boundaries:

$$y' = y, \ x' = x/(1 - \eta \cos k_0 y) \tag{2.2}$$

We shall seek the solution $U = (\psi, T)$ of the problem (1.1), (1.2) written in the new coordinates in the form

$$U = U_0 + \sum_{n=1}^{\infty} \varepsilon^n U_n \tag{2.3}$$

Assuming that $G - G_c = O(\varepsilon^2)$, we introduce the notation

$$G - G_c = \varepsilon^2 G_2 \tag{2.4}$$

The dimensionless fluid flow q is an independent parameter. Nevertheless, we shall first consider, in detail as in /6/, the case when the quantity q is small and together with $G - G_c$, of order ε^2

$$q = \varepsilon^2 q_2 \tag{2.5}$$

We will also assume that the wave number characterizing the curvature of the boundaries k_0 is close to k_c

$$k_0 - k_c = \varepsilon k_1 \tag{2.6}$$

Following /9/, we assume that the functions U_n depend on two spatial scales $y_0 = y'$, $y_1 = \varepsilon y'$, and several time scales $t_n = \varepsilon^n t$, and make the following substitution in the equations:

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y_0} + \varepsilon \frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial t} = \sum_{n=0}^{\infty} \varepsilon^n \frac{\partial}{\partial t_n} \tag{2.7}$$

We will write the relation connecting the parameters ε, η and G in the form

$$\eta = \sum_{n=1}^{\infty} \varepsilon^n \eta^{(n)}(G) \tag{2.8}$$

Substituting (2.3)-(2.8) into (1.1), (1.2) and taking (2.2) into account, we obtain a boundary value problem of the n-th order in ε, and find $\eta^{(n)}$ from the condition for it to be solvable.

To the first order in ε the solution has the form

$$U_1 = 2 \operatorname{Re} [a_1 (y_1, t_1, t_2, \ldots) u(x') \exp (ik_c y_0)], \ \eta^{(1)} = 0$$

where $u = (\varphi, \vartheta)$ is a function describing a neutral perturbation of a plane parallel flow in a layer with plane boundaries. To the second order we have

$$\eta^{(2)} = 0, \quad \frac{\partial a_1}{\partial t_1} = 0$$

Finally, the condition that the equations have a solution to the third order (for more detail see /5/), yields

$$I \frac{\partial a_1}{\partial t_2} = R \frac{\partial^2 a_1}{\partial y_1^2} + (JG_2 + iBq_2) a_1 - S |a_1|^2 a_1 + D\eta_3 \exp(ik_1 y_1) \tag{2.9}$$

$$B = -k_c \int_{-1}^{1} dx' \left\{ \frac{3}{4}(1 - x'^2) \left[-\overline{\varphi}_c \left(\frac{\partial^2 \varphi}{\partial x'^2} - k_c^2 \varphi \right) + \overline{\vartheta}_c \vartheta \right] - \frac{3}{2} \overline{\varphi}_c \varphi \right\}$$

where I, R, J, S, D are real constants obtained in /5/, and (φ_c, ϑ_c) is a solution of the conjugate linear problem. The amplitude equation (2.9) differs from one obtained earlier /6/ for the case of mixed convection, in the presence of the additional dispersion-type term $R\partial^2 a_1/\partial y_1^2$ on the right hand side. Its appearance is related to the fact that the amplitude a_1 depends not only on the time (as was assumed in /6/), but is also a slow function of the longitudinal coordinate $y_1 = \varepsilon y'$ (in this case a_1 has the meaning of the envelope of a wave packet with the carrier wave number k_c). Such a generalization makes possible the study of the stability of spatially periodic motions not only under perturbations with the same period, but also under non-periodic, general-type perturbations which have the form of Flocquet functions. We also note that the presence of the complex term $iBq_2 a_1$ in the amplitude equation of /6/ was connected with the peristaltic pumping caused by the wave-like motion of the layer boundaries. The appearance of the analogous term in the present paper is caused by the forced pumping of the fluid relative to the fixed curved boundaries.

Assuming, to be specific, that $Bq_2 > 0$, we transform the scale

$$Z = a_1 (S/Bq_2)^{1/2}, \ \tau = Bq_2 t_2/I, \ Y = y_1 (Bq_2/R)^{1/2}$$

to reduce equation (2.9) to the form

$$\frac{\partial Z}{\partial \tau} = \frac{\partial^2 Z}{\partial Y^2} + (\gamma + i) Z - |Z|^2 Z + \delta \exp(iK_0 Y) \tag{2.10}$$

$$\gamma = J\,(G - G_c)/Bq,\ K_0 = (k_0 - k_c)\,(R/Bq)^{1/2}$$
$$\delta = \eta_3 D S^{1/2}/(Bq_2)^{3/2}$$

Equation (2.10) is derived under assumption (2.5). In this case the time scales determining the growth of the perturbation amplitude and change of its phase (characterized by the increment and the frequency) are of the same order, and the equations are found to be most interesting. The case when $q \gg \varepsilon^2$ (strong pumping of fluid) corresponds to the limit $\gamma \to 0,\ \delta \to 0$ and can also be analysed with help of (2.10). If on the other hand $q \ll \varepsilon^2$, then the term containing q_2 vanishes from (2.9) and we have the case studied earlier in /5/.

3. Solutions of (2.10) of the form

$$Z = z\,(\tau)\exp(iK_0 Y) \tag{3.1}$$

where $z(\tau)$ satisfy the equation

$$dz/d\tau = (\Gamma + i)\,z - |z|^2 z + \delta,\ \Gamma = \gamma - K_0^2 \tag{3.2}$$

correspond to spatially peridoic solutions of the problem (1.1), (1.2) with period $2\pi/k_0$. Figure 1 shows schematically the parametric representation of Eq.(3.2), established in /6/. In the region between the lines 1 and 2 described by the formula

$$\Gamma = \Gamma_{\mp}(\delta),\quad \delta^2 = \frac{2}{27}\,[\Gamma_{\mp}(\Gamma_{\mp}^2 + 9) \pm (\Gamma_{\mp}^2 - 3)^{3/2}] \tag{3.3}$$

the equation has three stationary solutions, and one solution outside this region. The stationary solutions

$$z = X\exp(i\varphi) \tag{3.4}$$

are determined by the equations

$$\Gamma = X^2 \pm (\delta^2/X^2 - 1)^{1/2},\ \operatorname{tg}\varphi = -1/(\Gamma - X^2) \tag{3.5}$$

Moreover, a cycle exists within the cross-hatched region in Fig. 1. Boundary 3 of this region is described by the equation

$$\Gamma = \Gamma_0(\delta),\ \delta^2 = {}^1/_8\,\Gamma_0(\Gamma_0^2 + 4) \tag{3.6}$$

and boundary 4 was found in /6/ by numerical methods. The values of the coordinate δ for the points P_1, P_2, P_3 and P_4 are, respectively,

$$\delta_1 = (8\sqrt{3}/9)^{1/2} \approx 1.241,$$
$$\delta_2 = \sqrt{2} \approx 1.414,$$
$$\delta_3 \approx 1.269,\quad \delta_4 \approx 1.280$$

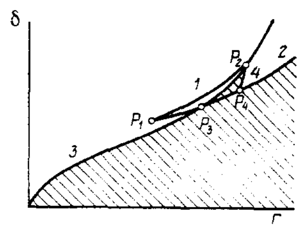

Fig. 1

4. We will now consider the stability of the spatially periodic motions. We shall deal with the stability of the stationary motions, and in Sect.5 with the stability of the oscillatory motions corresponding to the time-periodic solutions of (3.6).

Let us impose a small perturbation ζ on the basic solution $Z_0 = z\exp(iK_0Y)$, $z = X\exp(i\varphi)$. Substituting $Z = Z_0 + \zeta$ into (2.10) and linearizing over ζ, we obtain

$$\frac{\partial\zeta}{\partial\tau} = (\gamma + i)\,\zeta + \frac{\partial^2\zeta}{\partial Y^2} - X^2\,(2\zeta + \exp[2i(\varphi + K_0Y)]\,\bar\zeta) \tag{4.1}$$

Putting now

$$\zeta = a\exp[i\,(K_0 + K)\,Y + \lambda\tau] + \bar b\exp[i\,(K_0 - K)\,Y + \bar\lambda\tau] \tag{4.2}$$

we arrive at the homogeneous system of equations for the coefficients a and b. The condition that it has a solution, yields the following expression for the increment λ:

$$\lambda_\pm = \gamma - K_0^2 - 2X^2 - K^2 \pm [X^4 + (2K_0K - i)^2]^{1/2} \tag{4.3}$$

The formula generalizes (3.2) of /6/ where the case of $K = 0$ (the perturbations have the same period as the basic motion) was considered. We note that when $K_0 = 0$ $(k_0 = k_c)$, (4.3) becomes

$$\lambda_\pm = \gamma - 2X^2 - K^2 \pm (X^4 - 1)^{1/2} \tag{4.4}$$

We see that in this case we have $K = K_* = 0$ for the most dangerous perturbation, so that the investigation of the stability reduces to that carried out in /6/.

The complete computation of the region of stability of stationary solutions in the (γ, K_0^2) plane for a given δ, requires the determination of the sign of the real part of the increment

$$\operatorname{Re}\lambda_+ = \gamma - K_0^2 - 2X^2 - K^2 + \frac{1}{\sqrt{2}}[\Delta + (\Delta^2 + 16K^2K_0^2)^{1/2}]^{1/2} \tag{4.5}$$
$$\Delta = X^4 - 1 + 4K^2K_0^2$$

Here X^2 is a known function of δ, γ, and K_0^2 is defined implicitly by the first equation of (3.5).

Suppose that initially $\delta < \delta_1$. In this case the stationary solution is unique for any γ. We can find the asymptotic form of the boundary of stability for small K_0. Putting $4K^2K_0^2 \ll 1 - X^4$ we obtain from the condition $\partial \operatorname{Re}\lambda_+/\partial K = 0$ the quantity $K = K_*$ for the most dangerous perturbation

$$K_* = K_0(1 - X^4)^{-1/2} \tag{4.6}$$

The equation of the boundary of stability $\operatorname{Re}\lambda_+(K_*) = 0$ reduces to second order terms, to the form

$$\gamma = \Gamma_0 + \frac{3\Gamma_0^2}{4 + 3\Gamma_0^2}K_0^2 \tag{4.7}$$

where $\Gamma_0(\delta)$ is given by (3.6). The asymptotic form of the boundary of stability for large K_0 is also easily obtained

$$K_* = K_0,\ \gamma = 2\delta^2 K_0^{-4} \tag{4.8}$$

We see that for large $K_0 \infty k_0 - k_c$ the effect of external modulation on the stability of motion is weakened, and one of the wave numbers of the critical perturbation tends to k_c. A typical boundary of stability when $\delta < \delta_1$ constructed by numerical methods is shown in Fig.2 (the line 1; $\delta = 1$).

When δ becomes greater than δ_1, three stationary solutions appear in a certain region (γ, K_0^2). As was shown in /6/, one of them (the one with an intermediate value of X^2) is always unstable. The remaining two solutions have a region of stability with respect to the perturbations, with $K = 0$, and the solution with the largest value of X^2 is stable for any $\delta > \delta_1$ in the region $-\infty < \gamma < \Gamma_+$, while the solution with the smallest value of X^2 is stable when $\delta_1 < \delta < \delta_2$ in the region $\Gamma_- < \gamma < \Gamma_0$.

Figure 3 shows how the pattern of stability regions changes as δ increases in the case when $K \neq 0$, for both types of motion (the fragment *a* corresponds to $\delta = \delta_1$). Line *2* represents the boundary of stability of the mode with largest X^2, and lines *1* and *3* the mode with the smallest. The same notation is used in Fig. 4-6 which show the boundaries of the regions of stability for $\delta = 1.278$, $\delta = 1.304$, $\delta = 3.161$.

Note that the region of stability of motion with the smallest X^2 is not large, and vanishes when $\delta > \delta_2$. We also note that when $X^2 > 1$, the asymptote of the boundary of the region of stability (for the mode with maximum value of X^2) differs from (4.6), (4.7). Namely, the analysis of the dependence of the increment $\lambda_+(K)$ within the limits $4K^2K_0^2 \ll X^4 - 1$, shows that when

$$K_0 < K_m = \left[\frac{1}{2}X^{-4}(X^4 - 1)^{3/4}\right]^{1/2} \tag{4.9}$$

the boundary of stability of the stationary solutions is determined by the perturbations with $K_* = 0$, and coincides with the boundary of their existence $\gamma = \Gamma_+(\delta) + K_0^2$, while when $K_0 > K_m$, the most dangerous perturbations are those with

$$K_* = (X^4 - 1)(X^4 + 4)^{-1/2}(K_0 - K_m)^{1/2}K_m^{-1/2}$$

Thus we have, in the interval $K_0 < K_m$, a singular "capture of the wave number" where the resonant harmonic tunes to the external perturbation.

For fairly large δ (slow oscillations of the fluid) the boundary of stability takes the form shown in Fig.6 (curve *2*; $\delta = 3.161$). The characteristic feature of this graph consists of the existence of the interval of wave numbers $K_1 < K_0 < K_2$ within which the stationary motion first loses its stability as γ increases (point *A*), then regains it (point *B*) and finally loses it again (point *C*).

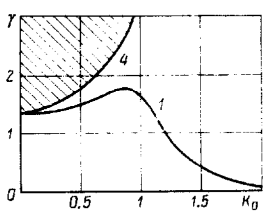

Fig. 2

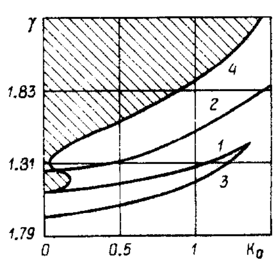

Fig. 4

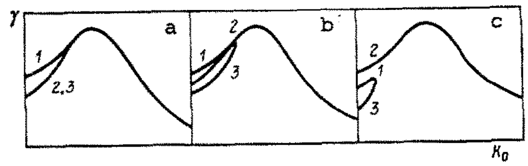

Fig. 3

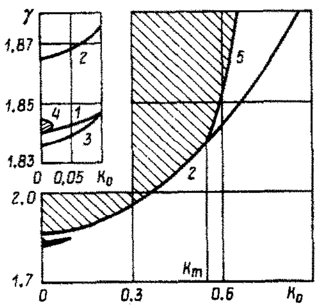

Fig. 5

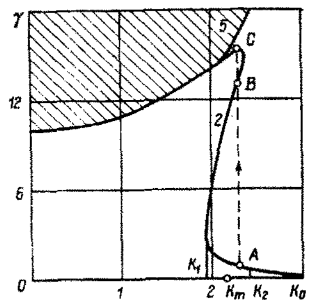

Fig. 6

5. Let us now consider the stability of non-stationary convection modes corresponding to the time-periodic solutions of (3.2). The evolution of an infinitesimal perturbation is described, as before, by (4.1), but in this case $z = X\exp(i\varphi)$ is a periodic functions of time. Let us write the perturbation in the form

$$\zeta = a(\tau)\exp[i(K_0 + K)Y] + b(\tau)\exp[i(K_0 - K)Y] \qquad (5.1)$$

The functions $a(\tau), b(\tau)$ satisfy the following system of equations with peridoic coefficients:

$$da/d\tau = [\gamma - (K_0 + K)^2 - 2X^2 + i]\,a - X^2\exp(2i\varphi)\,\bar{b} \qquad (5.2)$$
$$d\bar{b}/d\tau = [\gamma - (K_0 - K)^2 - 2X^2 - i]\,\bar{b} - X^2\exp(-2i\varphi)\,a$$

whose fundamental solutions have the form of Flocquet functions

$$f(\tau) = (a(\tau), \bar{b}(\tau)),\ f(\tau) = g_+(\tau)\exp\lambda_+\tau + g_-(\tau)\exp\lambda_-\tau \qquad (5.3)$$
$$g_\pm(\tau) = g_\pm(\tau + T),\ \mathrm{Re}\,\lambda_- \leqslant \mathrm{Re}\,\lambda_+$$

where T is the period of the functions z. The condition of stability of periodic solution is $\mathrm{Re}\,\lambda(K) \leqslant 0$ for all K.

The boundary of the region of stability of oscillatory motions is obtained most simply when $\delta \ll 1$. In this case natural osillations develop when $\gamma > \Gamma_0(\delta) + K_0^2$, $\Gamma_0(\delta) \doteq 2\delta^2$, and are described by the formula /6/

$$z = (\Gamma - 2\delta^2)^{1/2}\cos\tau + i[\delta + (\Gamma - 2\delta^2)^{1/2}]\sin\tau,\ \Gamma = \gamma - K_0^2$$

For $\lambda_\pm$ we find

$$\lambda_\pm = -(\gamma - 2\delta^2 - K_0^2 + K^2) \pm [(\gamma - 2\delta^2 - K_0^2)^2 + 4K_0^2K^2]^{1/2}$$

and this implies that the cycles are stable when $\gamma > 2\delta^2 + 3K_0^2$.

System (5.2) was integrated numerically for finite δ. The increment of the most rapidly increasing mode was calculated using the formula

$$\mathrm{Re}\,\lambda_+ = \lim_{\tau\to\infty}\frac{1}{2T}\ln\frac{[|a(\tau + T)|^2}{|a(\tau)|^2}$$

When $\delta < \delta_1$, the form of the region of stability of the cycles is qualitatively the same as in the case when $\delta \ll 1$ Fig.2, curve 4; $\delta = 1$). As δ approaches δ_4 (see Fig. 1), the boundary of the region of stability takes the form shown in Fig. 4 (line 4; $\delta = 1.278$). When $\delta_3 < \delta < \delta_2$, Fig. 1 shows that the cycle exists in two intervals of variation of Γ, separated by a region (between the curves 4 and 2) within which no oscillatory mode exists. Similarly, two regions of stability of the cycles exist (Fig. 5; $\delta = 1.304$), one of which, bounded by curve 4, is adjacent to the region of stability of the stationary mode with minimum amplitude, and the other (boundary curve 5) to the region with maximum amplitude. Note that the lines 2 and 5 intersect when $K_0 = K_m$ (see formula (4.9)), so that the oscillatory motions $K_0 < K_m$ are stable everywhere within the domain in which they exist.

When $\delta > \delta_2$, a unique boundary of the region of the cycle stability remains (Fig. 6, curve 5; $\delta = 3.161$) with a break when $K_0 = K_m$. In the limit when $\delta \gg 1$, we use expansions in terms of the parameter δ^{-2} to obtain the expression for the increment

$$\lambda_{\pm} = -(\gamma - K_0^2 + K^2) \pm [(\gamma - K_0^2)^2 + 4K_0^2K^2]^{1/2}$$

and this implies that the boundary of the region of stability of an oscillatory region has the form $\gamma = 3K_0^2$ when $K_0^2 > \delta^2/2$. When $K_0^2 < \delta^2/2$, the boundary coincides with the boundary of existence of the cycles $\gamma = \delta^2 + K_0^2$.

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Translated by L.K.

PMM U.S.S.R., Vol.48, No.6, pp. 688-694, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

WEAKLY SUPERCRITICAL DISSIPATIVE STRUCTURES ON CURVED SURFACES *

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Cellular, low amplitude structures appearing at cylindrical and spherical fronts of gaseous combustion and laser evaporation are described. In the case of a spherical front all these structures are found to be unstable. When the cylindrical front of gaseous combustion is expanded, we must expect the quasi one-dimensional structure homogeneous with respect to the ignorable coordinate to be replaced by a parquet-like pattern of rectangular cells, and later to reach a non-stationary regime. On the cylindrical front of laser evaporation the quasi one-dimensional structure of maximum amplitude is globally stable.

The best known hydrodynamic example of a kinetic problem connected with the formation of dissipative structures i.e. thermodynamically non-equilibrium stationary structures appearing as a result of the development of aperiodic instability in a spatially homogeneous state, are Benard cells /1,2/. New problems of this kind are connected with the instability of plane fronts of laser evaporation of condensed material, and of gaseous combustion /3—5/. The instability is aperiodic and appears at finite values of the wave number of the perturbation representing curvature of a plane front. The development of the instability leads to the formation of a stationary, periodically curved front /3/.

The purpose of this paper is to investigate such structures and their stability on cylindrical and spherical surfaces, and this corresponds to

*Prikl. Matem. Mekhan., 48, 6, 942-949, 1984